Topology in physics 2019, exercises for lecture 4

- The hand-in exercises are exercises 1 and 3.
- Please hand it in electronically at `topologyinphysics2019@gmail.com` (1 pdf, readable!)
- Deadline is Wednesday March 6, 23.59.
- Please make sure your name and the week number are present in the file name.

⋆Exercise 1: Gauge fields or field strengts?

Use the exact sequence

$$0 \xrightarrow{f_0} H^1_{\mathrm{dR}}(M) \xrightarrow{f_1} \Omega^1(M)/d\Omega^0(M) \xrightarrow{f_2} \Omega^2_{\mathrm{cl}}(M) \xrightarrow{f_3} H^2_{\mathrm{dR}}(M) \xrightarrow{f_4} 0. \quad (1)$$

to show that in a generic situation where neither of the two cohomology groups is trivial, one has that

$$(\Omega^1(M)/d\Omega^0(M)) \times H^2_{\mathrm{dR}}(M) \cong \Omega^2_{\mathrm{cl}}(M) \times H^1_{\mathrm{dR}}(M) \quad (2)$$

Thus, also in general, 1-cohomology adds field configurations that can not be described by the field strength (as in the Aharonov-Bohm effect), whereas 2-cohomology adds field configurations that can not be described by a single 1-form gauge field (as in the Dirac monopole).

If space-time is $T^2 \times \mathbb{R}^2$ (T^2 is the two-torus), which description contains more information – the description in terms of gauge fields, or the description in terms of field strengths?

Exercise 2: Getting familiar with Yang-Mills theory

a. From the definition of the Yang-Mills field strength, $[D_\mu, D_\nu]\phi^i(x) = -iF^a_{\mu\nu}(x)T_a\phi(x)$, show that $F^a_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - if^a{}_{bc}A^b_\mu(x)A^c_n u(x)$.

b. Show that in form notation, this result can be written as $F = dA - iA \wedge A$.

c. In Maxwell theory, how do the previous two results simplify?

d. Show that the Yang-Mills action is invariant under local Lie group transformations.

e. Show that the Jacobi identity for Lie algebras, $[T_a, [T_b, T_c]] + \text{cycl.} = 0$ implies that $[D_\mu, [D_\nu, D_\rho]] + \text{cycl.} = 0$. ("+ cycl." means that terms with cyclic permutations of the indices must be added in the identity.)

f. Write the result of (e) as an identity for the field strength components, and show that this identity can be rewritten as the Bianchi identity $D_\mu(\star F)^{\mu\nu} = 0$.

$\star$Exercise 3: A first encounter with Chern-Simons theory

Just like for Maxwell theory, one might be tempted to write down a different action for the Yang-Mills gauge field, in which no Hodge star appears:

$$S = \int_M \mathrm{Tr}\ (F \wedge F). \tag{3}$$

Show that the integrand in this action is exact:

$$\mathrm{Tr}\ (F \wedge F) = d\mathrm{Tr}\ (A \wedge dA - \frac{2}{3} iA \wedge A \wedge A). \tag{4}$$

If our manifold M has a boundary, $\delta M = B$, use this to rewrite the above action as an integral on the boundary. Also, compute the Euler-Lagrange equation of motion for this action.

The theory described by this action is called *Chern-Simons theory*; it will play an important role in some of the later lectures.